Orbifolds

How to count every possible symmetry

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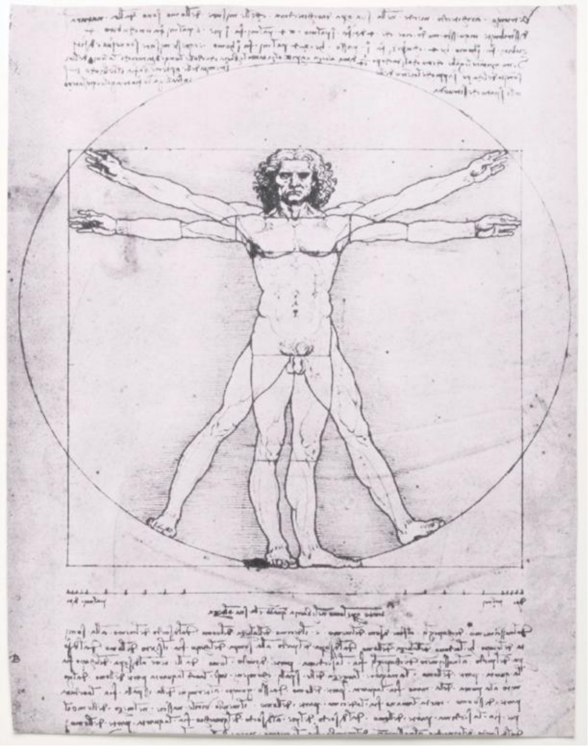
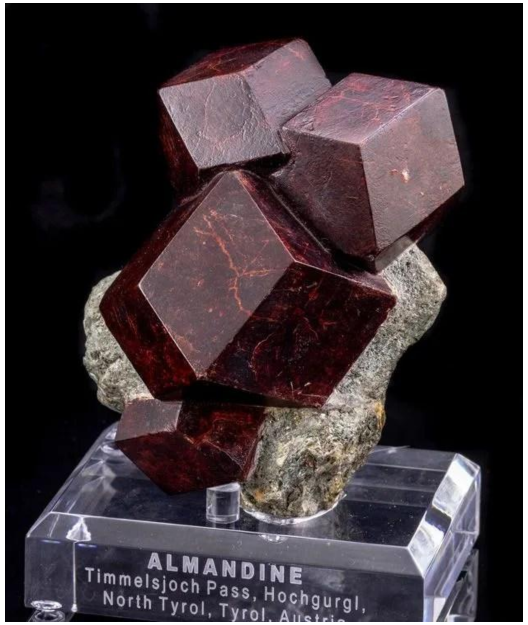
Graduate Student Seminar, Dartmouth College,

Spring 2025

1. What is symmetry / notation
2. How to enumerate symmetries
3. Why 1 – what does change mean
4. Why 2 – topology (brief)
5. Conclusion

What is symmetry?





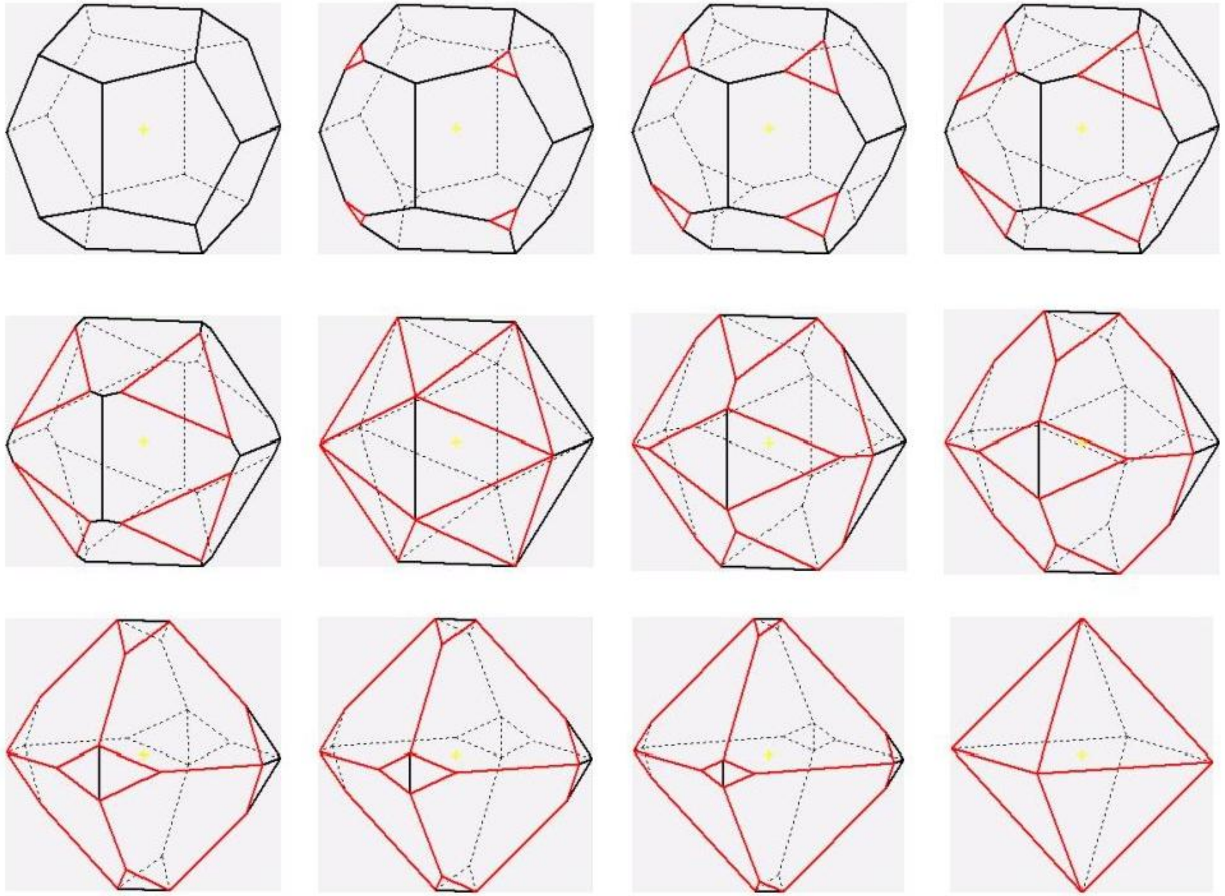
Each object (or subset X of $\mathbb{R}^n$) has a set of symmetries: distance preserving functions $f: \mathbb{R}^n \to \mathbb{R}^n$ which take that object to itself, that is, $f(X) = X$.

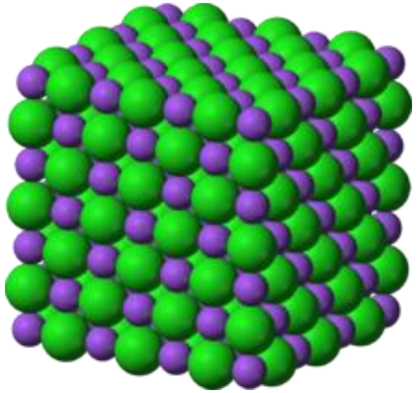
Central question: How many different groups are possible?

Crystals

- A crystal is any substance with a repeating molecular pattern (a **lattice**).
- Crystals form by molecules accumulating on this lattice, its symmetries shape those of the crystals.

https://www.mindat.org/article.php/1140/Crystal+Combinations+in+the+Isometric+System



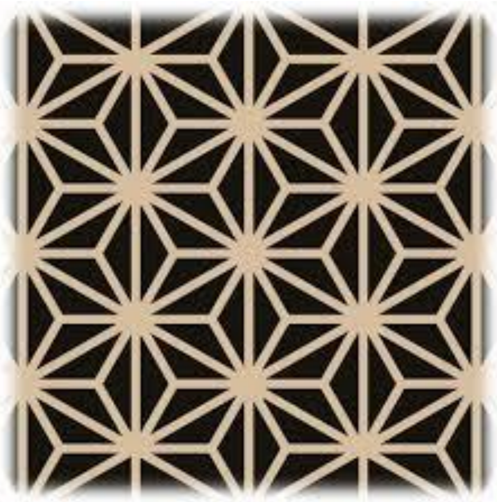


Depending on where these faces terminate, and many other environmental factors, a large variety of shapes can occur.

A great example of crystal habits - Pyrite

African Basket

Ukrainian Pysanky



Japanese Kumiko

Wall tiling in the Alhambra, Spain

Asian sand Mandalas

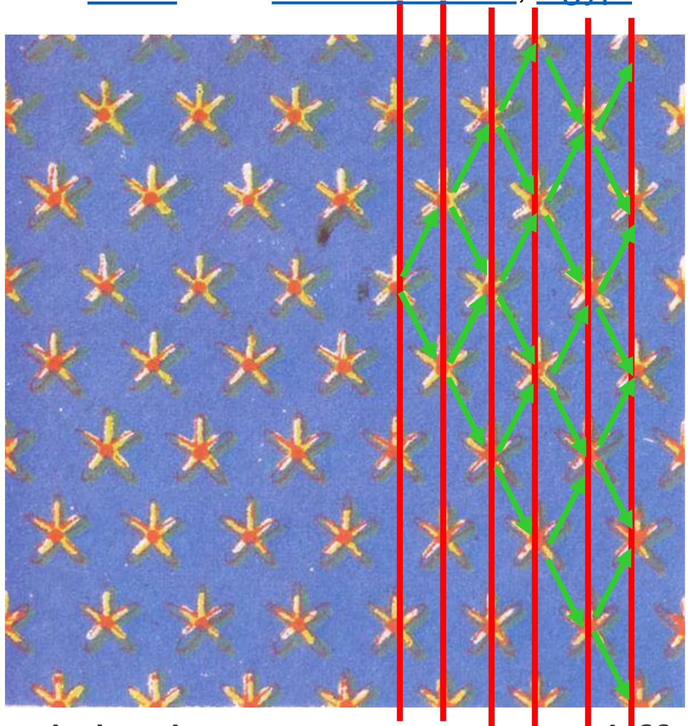
Navajo Weaving

Woven bamboo

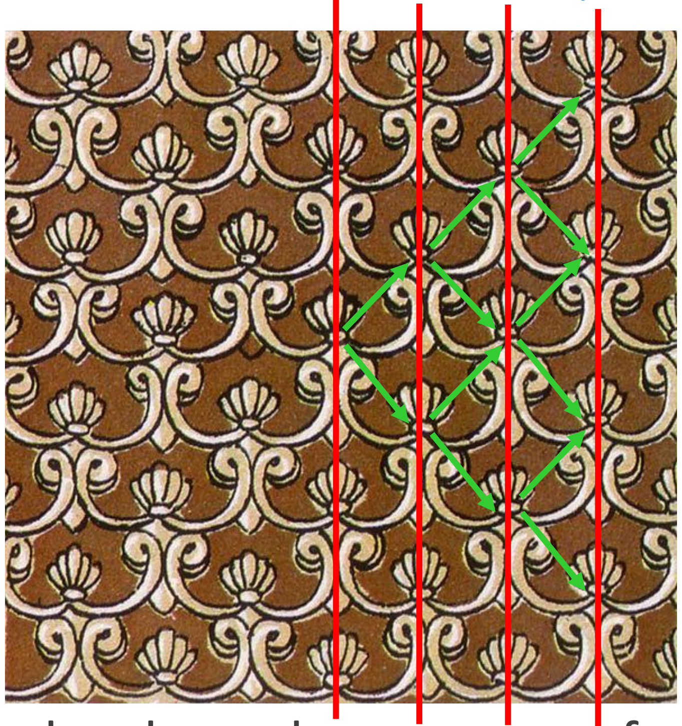
Different pattern, same symmetries

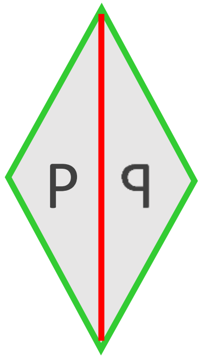
To classify all possible groups of symmetries, we ignore the designs / motifs.

Dado from Biban el Moluk, Egypt



Bronze vessel in Nimroud, Assyria





Both patterns are made by translating the same **translation cell**.

While these may appear different, they have the same set of symmetries.

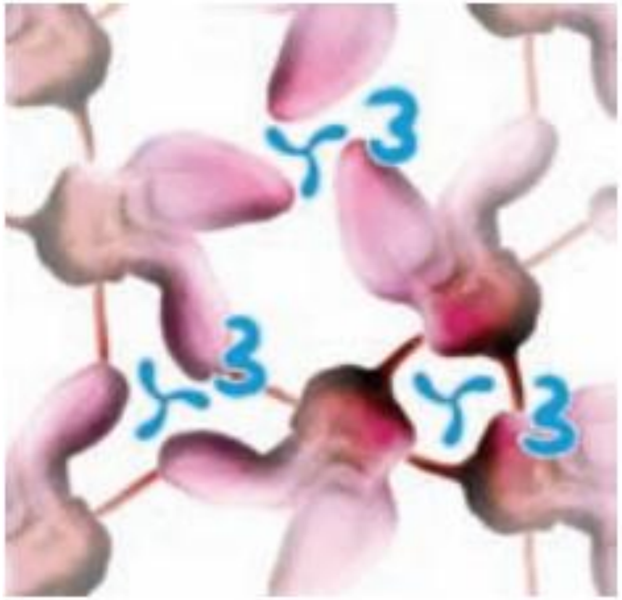
(We allow ourselves to stretch, rotate, and slide the plane when identifying the groups.)

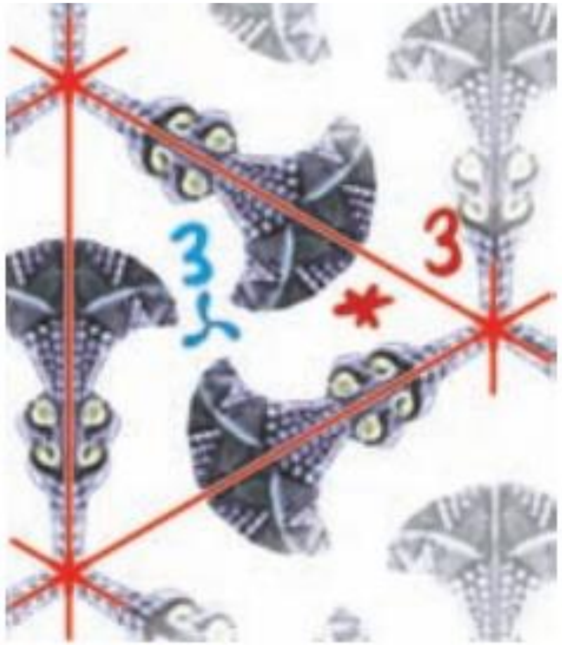
Kaleidoscopes and Gyrations

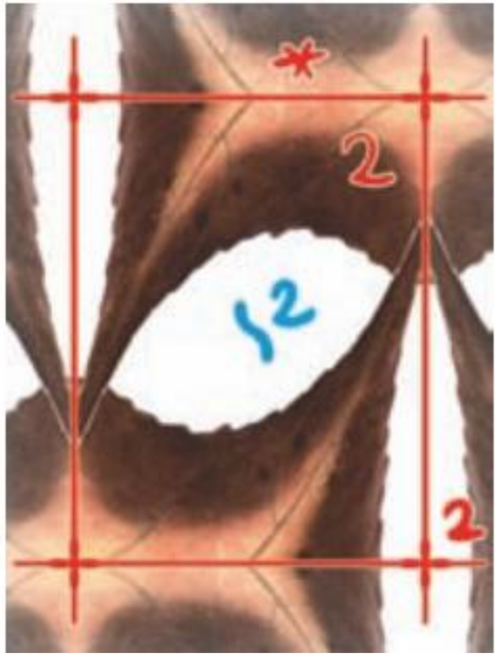
mirrors rotations

We can describe a pattern by identifying its symmetries with a signature:

***632** **333**





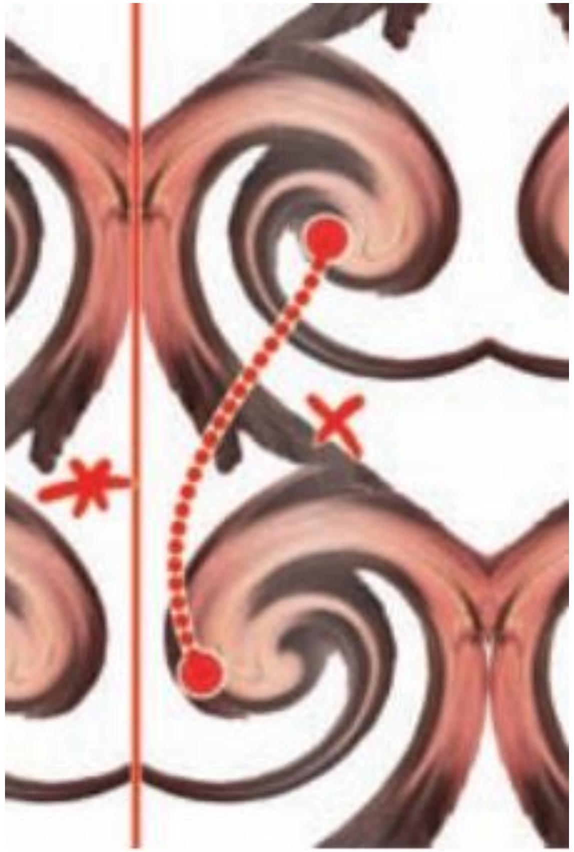


3*3 **2*22**

We use one number for each distinct point (one cannot be mapped to the other by a symmetry of the pattern).

Miracles and Wonders

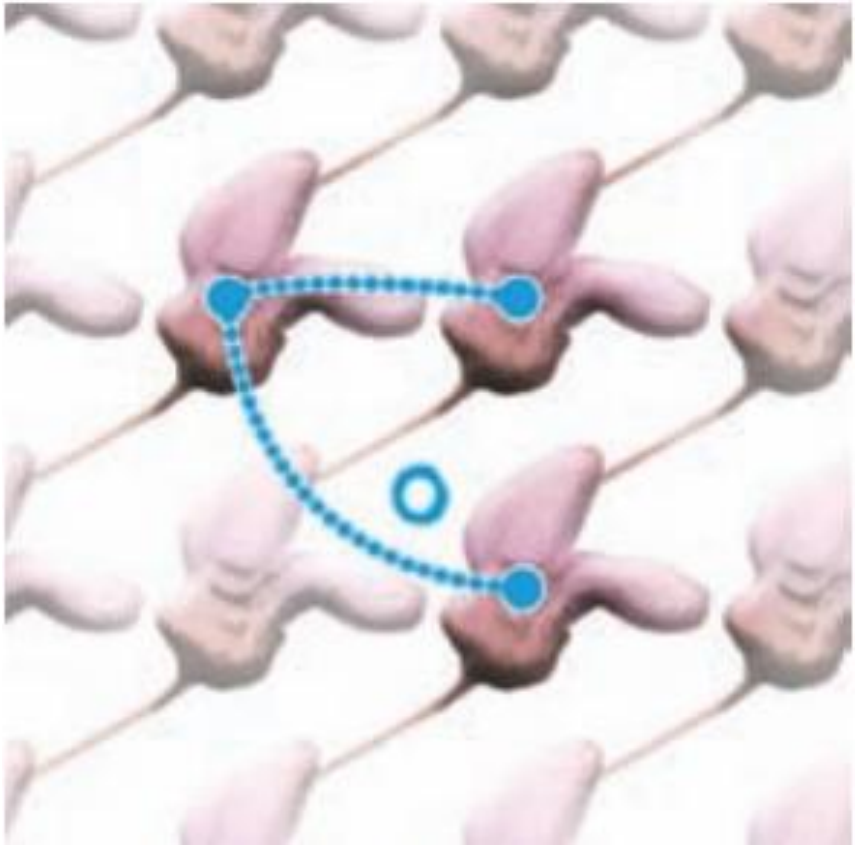
glides translations



*x

Mirrorless-crossing (miracle): a path between mirror images without crossings a mirror line.

Wandering (wonder-ring): a path between images not created by any previous operation (comes in pairs).



o

The Magic theorem

wonders	gyrations	kaleidoscopes	miracles
o...o	AB...C	*ab...c*de...f...	$\times\ldots\times$

Symbol	Cost (\$)	Symbol	Cost (\$)
o	2	* or $\times$	1
2	$\frac{1}{2}$	2	$\frac{1}{4}$
3	$\frac{2}{3}$	3	$\frac{1}{3}$
4	$\frac{3}{4}$	4	$\frac{3}{8}$
5	$\frac{4}{5}$	5	$\frac{2}{5}$
6	$\frac{5}{6}$	6	$\frac{5}{12}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	$\frac{N-1}{N}$	N	$\frac{N-1}{2N}$
∞	1	∞	$\frac{1}{2}$

Theorem: All plane repeating patterns have total cost \$2.

*632	$1+\frac{5}{12}+\frac{1}{3}+\frac{1}{4}=2$
333	$\frac{2}{3}+\frac{2}{3}+\frac{2}{3}=2$
3*3	$\frac{2}{3}+1+\frac{1}{3}=2$
2*22	$\frac{1}{2}+1+\frac{1}{4}+\frac{1}{4}=2$
*x	$1+1=2$
o	$2=2$

Enumeration of Planar symmetries

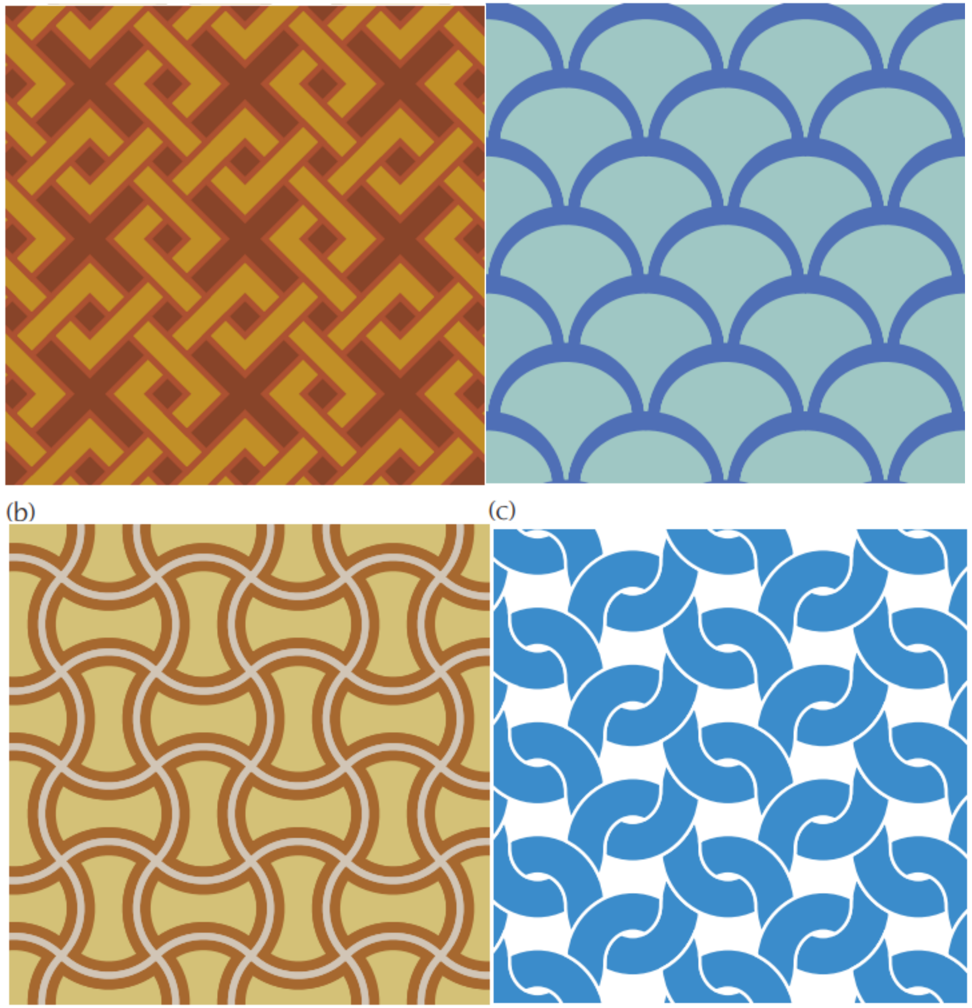
If only blue types are used: Each costs less than 1, so there must be at least three (**632**, **442**, or **333**). The smallest cost per symbol is ½ for 2, giving **2222**. A wonder ring alone costs 2, giving **o**.

The * costs 1, so we can use at most two to use red symbols. Since cost half as much as the corresponding blue symbol, we can add a * and make any of these entirely red, or add two *.

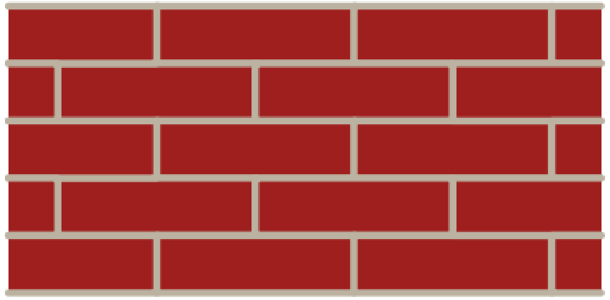
Lastly, for no change in cost we can *demote* ***nn** to **n***, or * to **x**.

$$\begin{array}{ccccc} \ast 632 & \ast 442 & \ast 333 & \ast 2222 & \ast\ast \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ & 4\ast 2 & 3\ast 3 & 2\ast 22 & \ast\times \\ & & & \downarrow & \downarrow \\ & & & 22\ast & \times\times \\ & & & \downarrow & \\ & & & 22\times & \end{array}$$

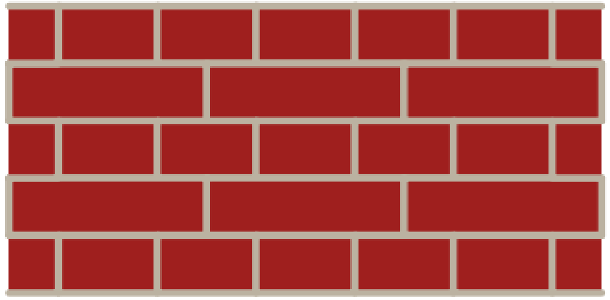
Examples



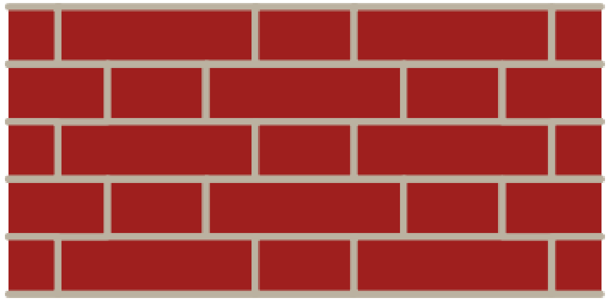
(b) (c) (d) (e)



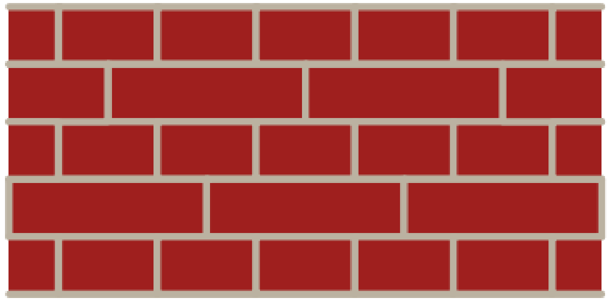
(a) Running bond



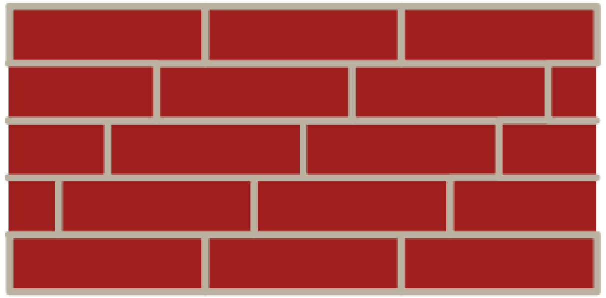
(b) English bond



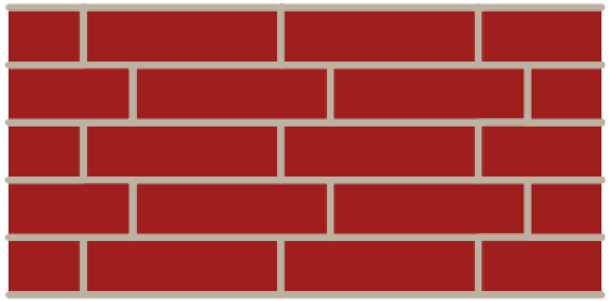
(c) Flemish bond



(d) Dutch bond

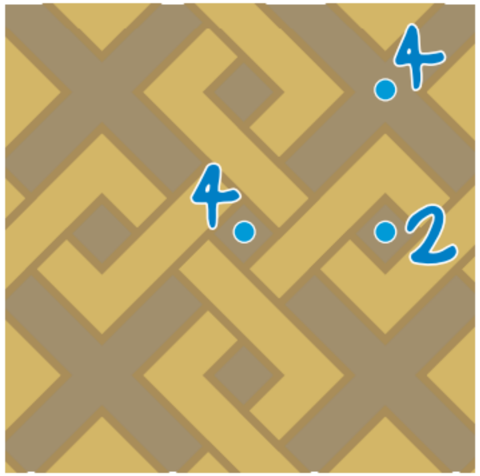


(e) Spiral bond

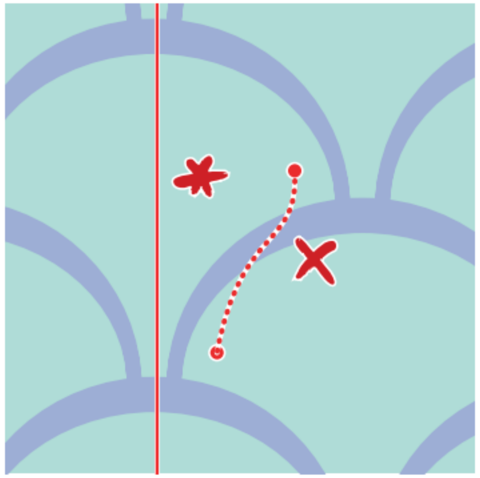


(f) Zigzag running bond

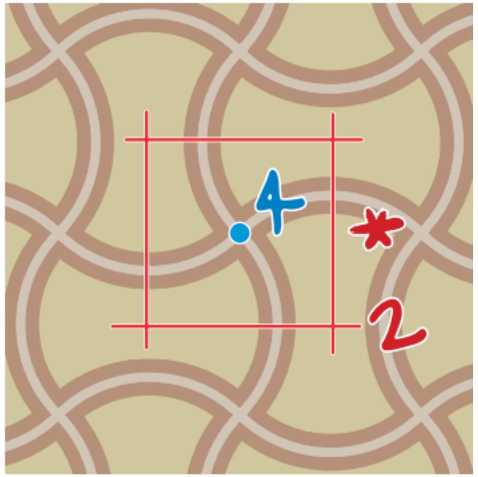
Examples



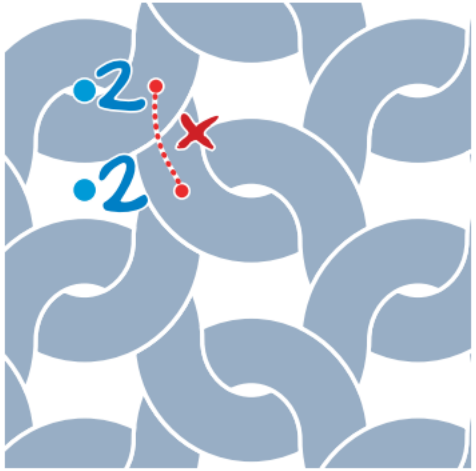
(b) 442



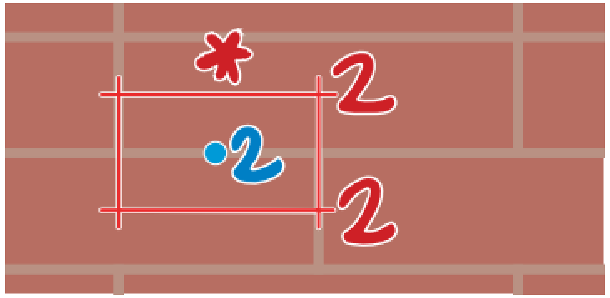
(c) *×



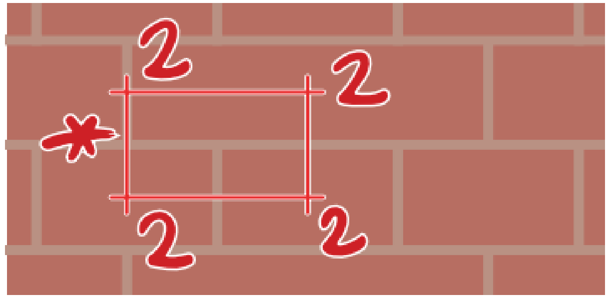
(d) 4*2



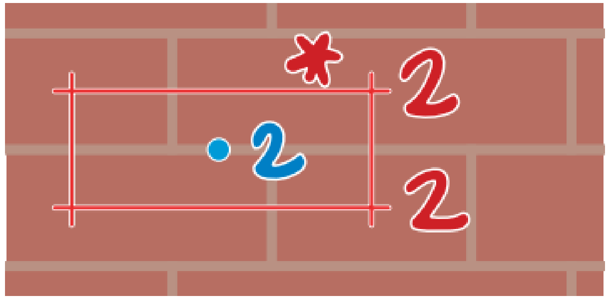
(e) 22×



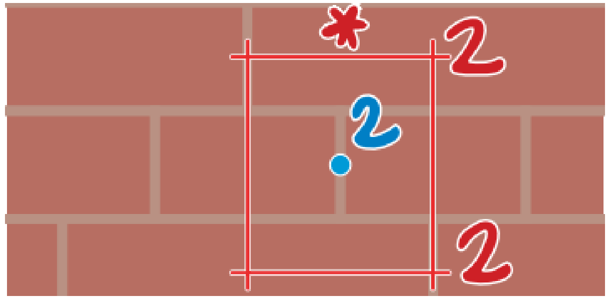
(a) Running bond has type 2*22



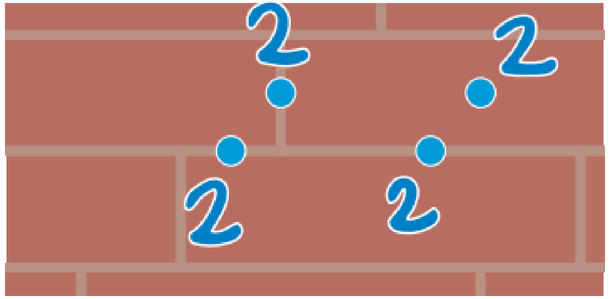
(b) English bond has type *2222



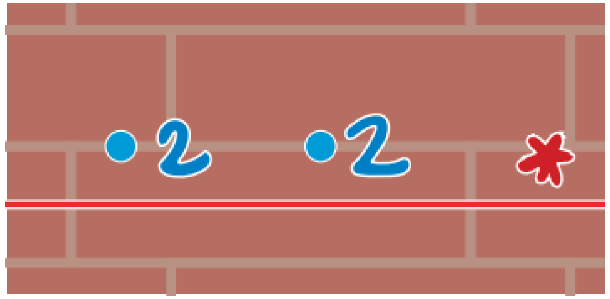
(c) Flemish bond has type 2*22



(d) Dutch bond also has type 2*22



(e) Spiral bond has type 2222



(f) Zigzag running bond has type 22*

Activity: identifying some symmetries

1. Mark any kaleidoscopes in red. If there are mirror lines, restrict attention to one of the regions into which they cut the plane. Put a red $*$ near any one kaleidoscope; then, find just one corner of each type (as in Chapter 2), and write the numbers of mirrors through each of these corners, also in red.

2. Look for gyration points. In blue, mark just one gyration point of each type with a spot and its order.

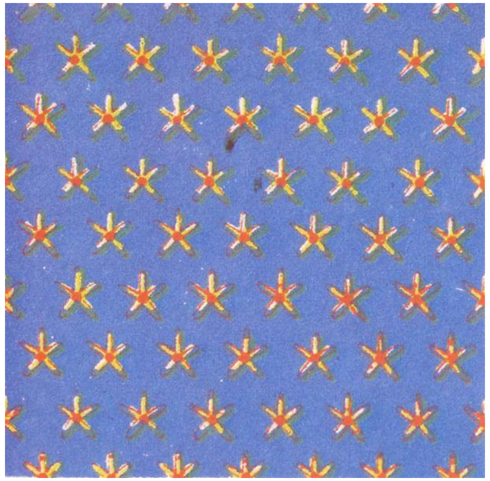
3. Are there miracles? Can you walk from some point to a copy of itself without ever touching a mirror line? If so, a miracle has occurred. Mark just one such path with a broken red line and a red cross nearby.

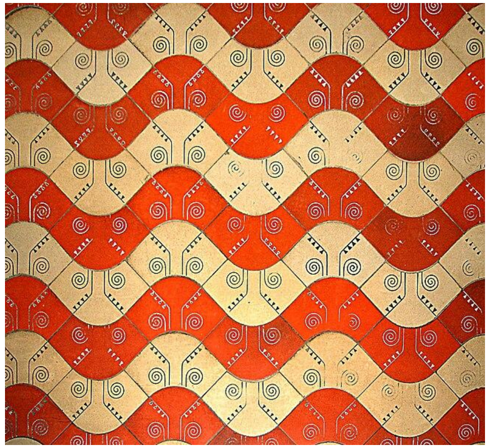
4. Is there a wonder? If you've found none of the above, then there is: mark it with a blue wonder-ring.

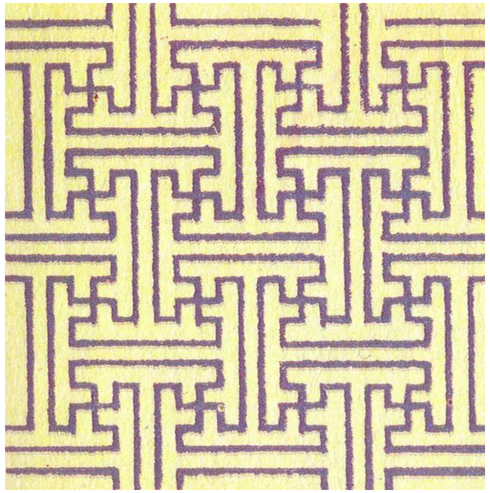
$*632$	$*442$	$*333$	$*2222$	$**$
			$2{*}22$	$*\times$
	$4{*}2$	$3{*}3$	$22*$	$\times\times$
			$22\times$	
632	442	333	2222	$\circ$

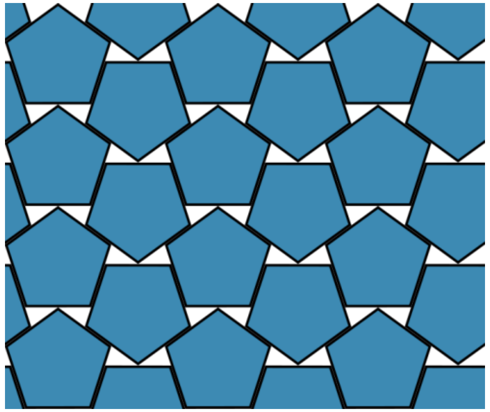
- If two features are the same you must only mark one of them;
- sometimes it helps to label gyration points before labeling kaleidoscopes.
- Be sure there aren't any mirror lines inside the region bounded by a kaleidoscope,
- don't forget that gyration points never lie on mirror lines!

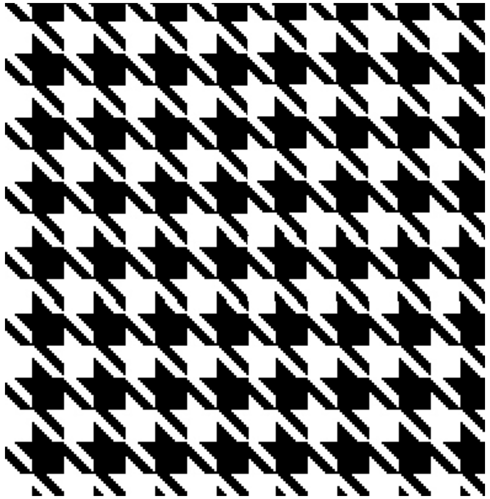
Match the Symmetries!



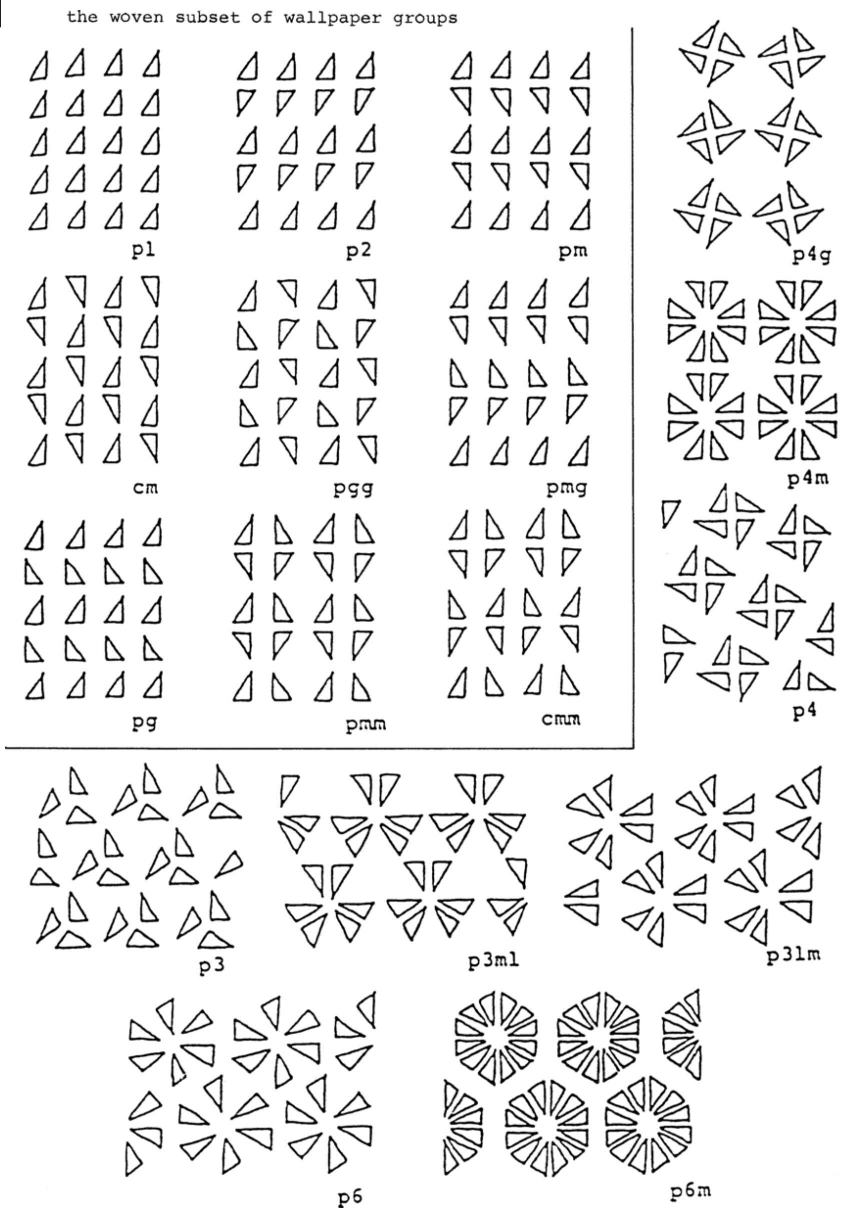


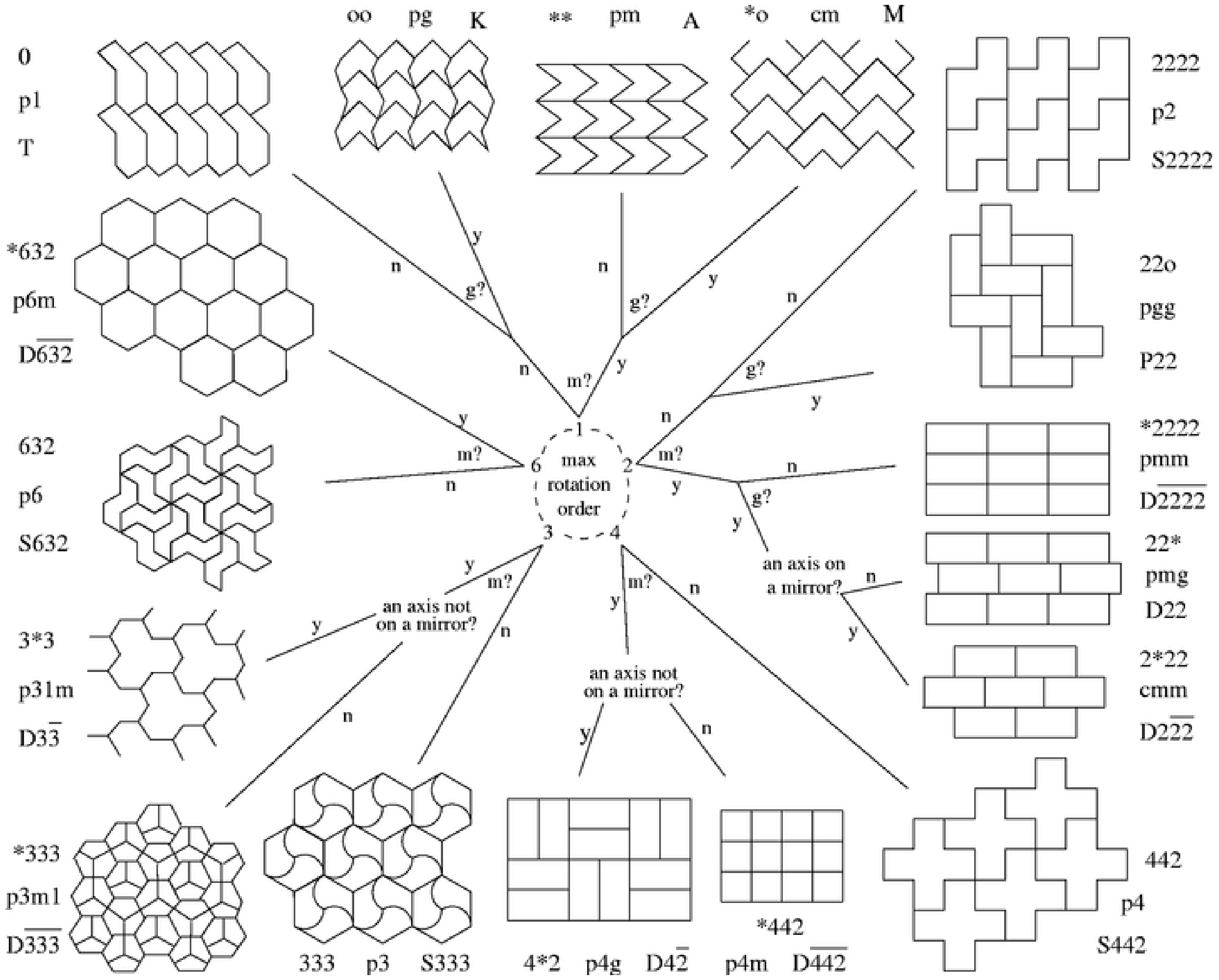






All 17 wallpaper groups





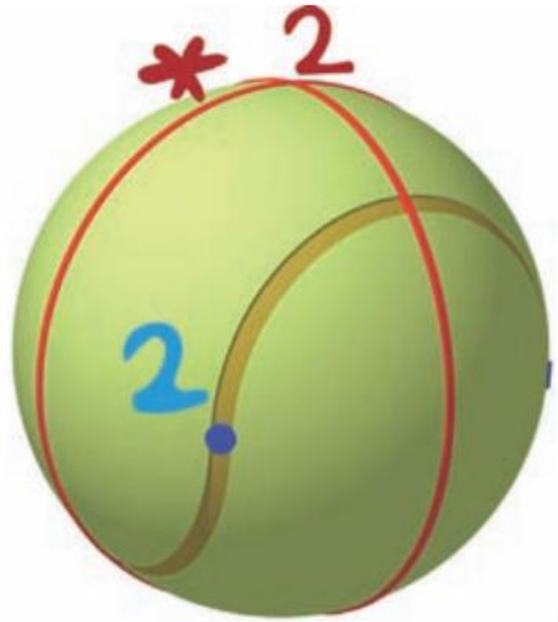
https://www.math.toronto.edu/drorbn/Gallery/Symmetry/Tilings/Sanderson/index.html

"Katachi U Symmetry in the Ornamental Art of the Last Thousands of Years of Eurasia" by Szaniszlo Berczi

3D Objects (Spheres!)

We project the symmetries of an object to the **Celestial Sphere**,

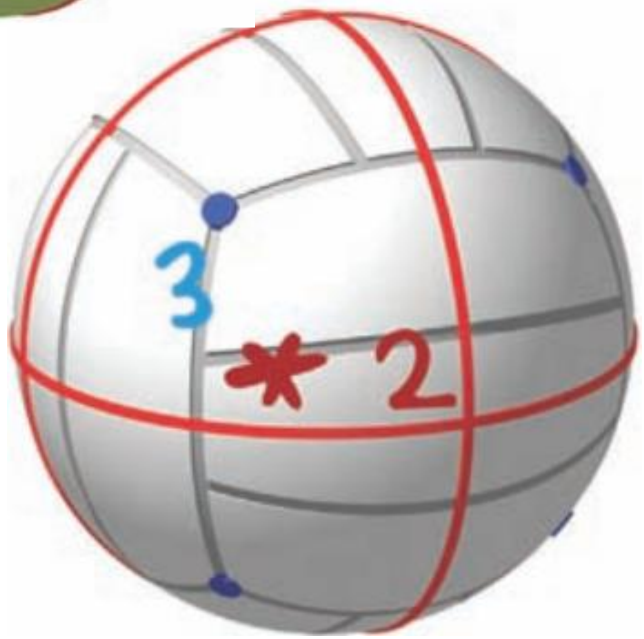
*22



2*2

Amazingly, the same operations work!

3*2



Theorem: The signature of a spherical pattern costs exactly $2 - \frac{2}{g}$, where g is the total number of symmetries.

All 14 spherical patterns

∗532	∗432	∗332	∗22N	∗MN
		3∗2	2∗N	N∗
				N×
532	432	332	22N	MN

(the types MN and ∗MN only exist if M = N. The other cases cause no problem, p57)

Examples

532

22N

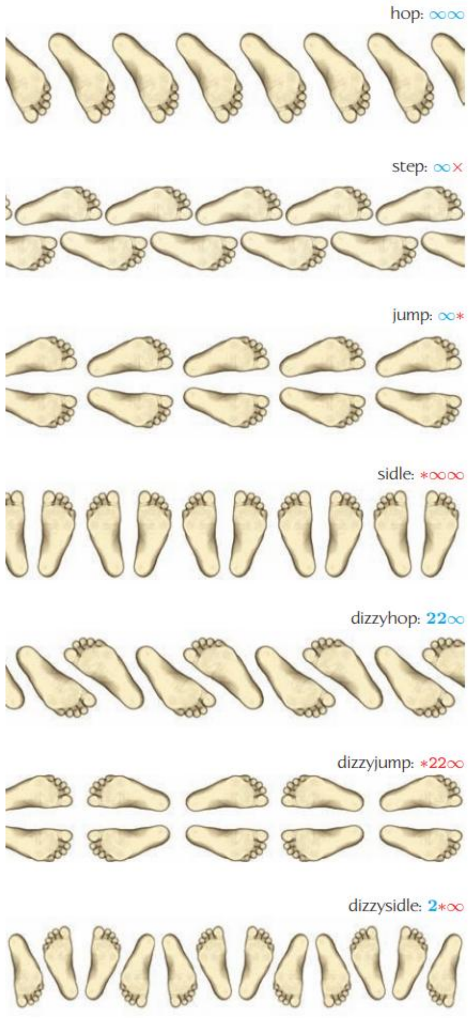
Temari - traditional Japanese embroidered balls

Sculptures by Bathesba Grossman

332

532

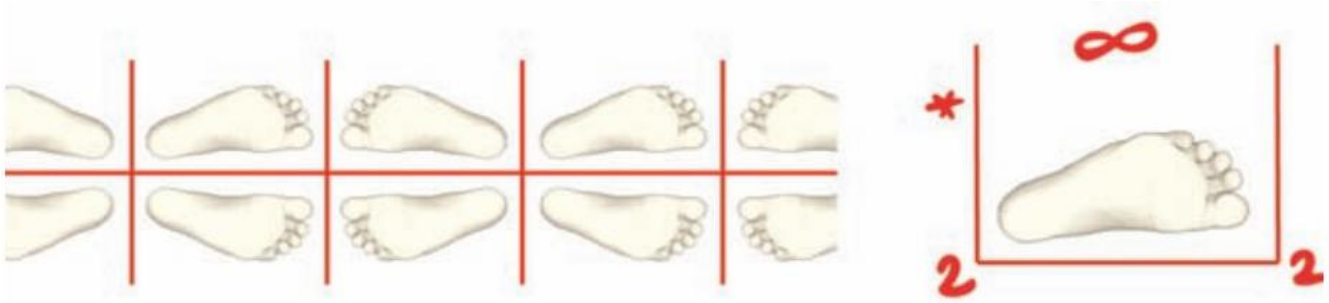
The 7 Frieze groups (one translation)



Theorem: The signatures of frieze patterns are precisely those that contain an ∞ symbol and cost exactly 2.

They are obtained by replacing an N in a spherical symmetry by infinity.

*22N



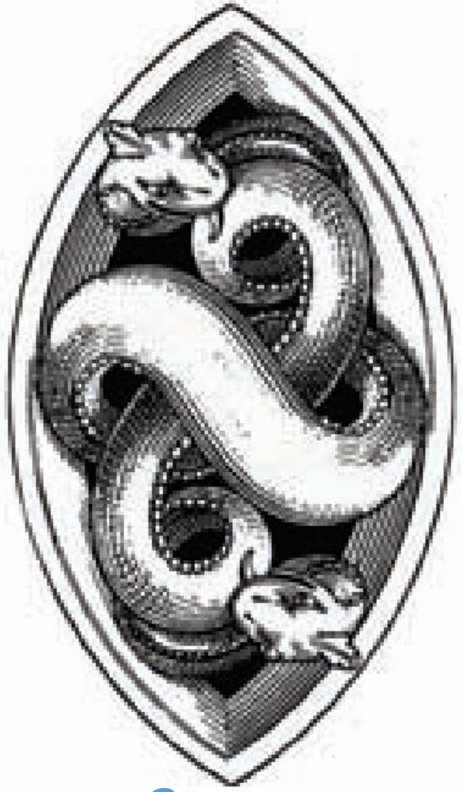
*22∞

Rosette groups (no translations)

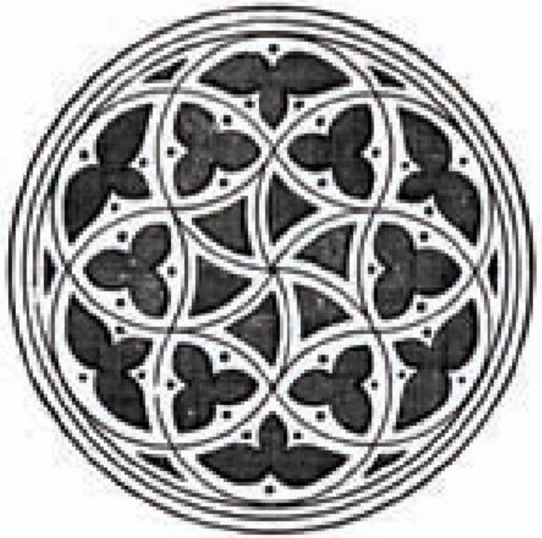
Finite plane patterns only contain rotations and mirrors. They are *N· and M· where · indicates they fix a point.

*4·

4·



2·



6·

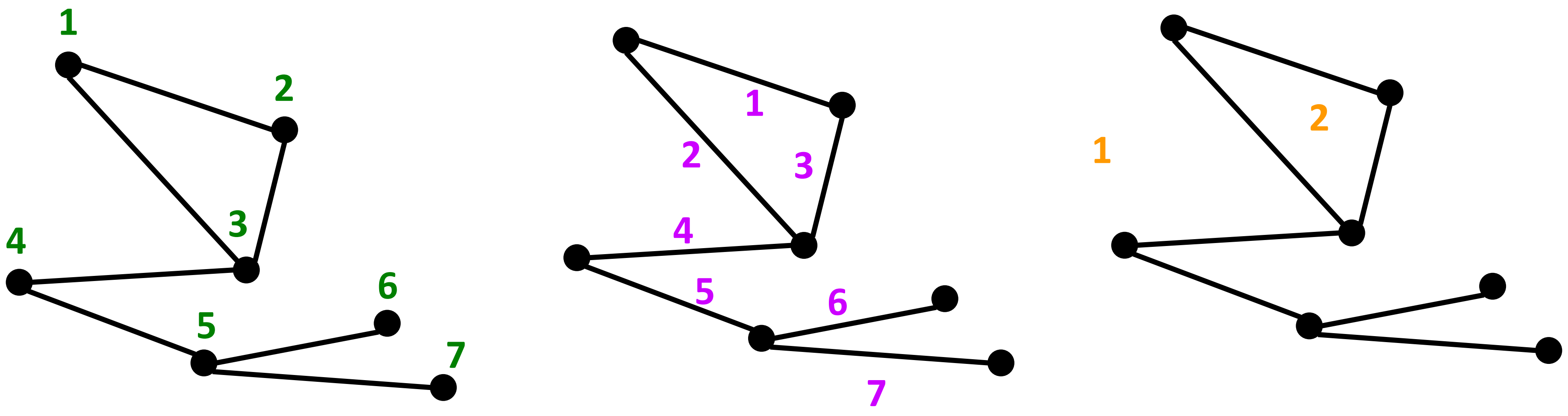
We identified four fundamental symmetries and used their associated costs to enumerate all possible groups of planar, spherical, frieze, and rosette symmetries.

But where does this theorem come from?

Graphs and the Euler Characteristic

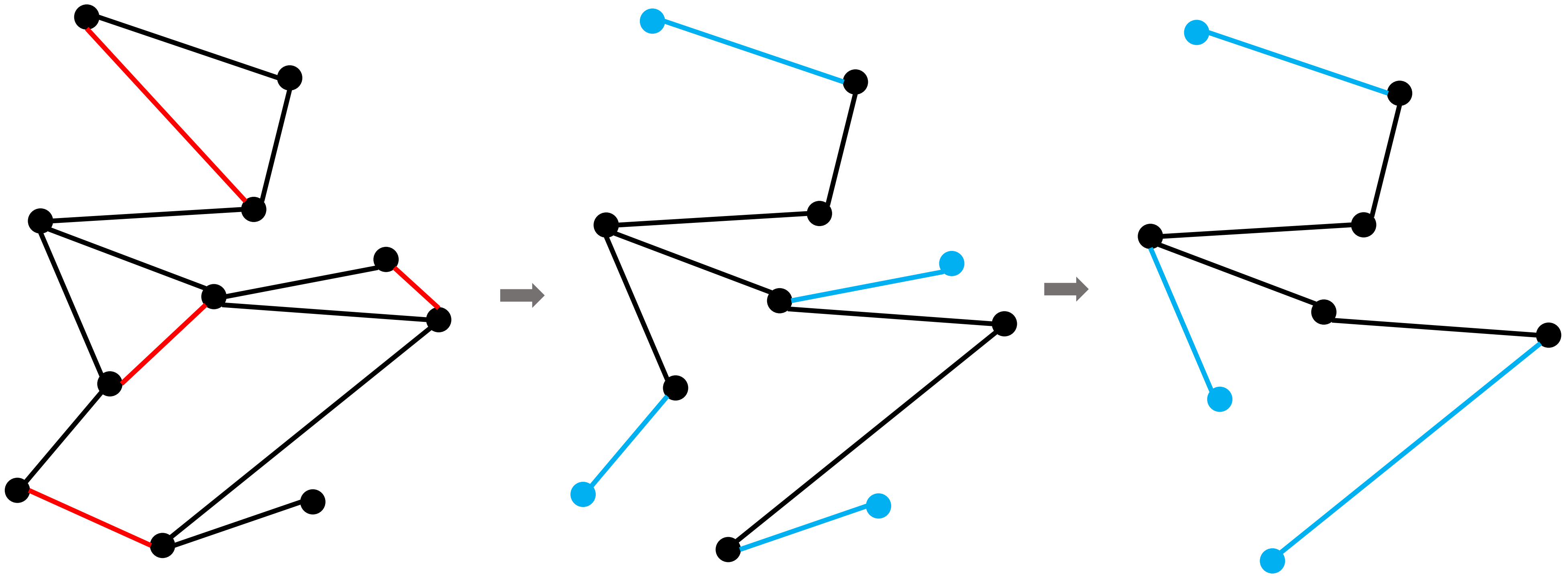
- A **graph** consists of vertices (dots) and edges (lines) between them.
- If we count the number of vertices V, edges E, and faces F (regions enclosed by edges) for a planar graph, we always find

$$V - E + F = 2 \qquad \text{(the \textbf{Euler Characteristic})}$$



$$7 - 7 + 2 = 2$$

Proof



Pop all the faces.

$$V - E + F = V - (E - 1) + (F - 1)$$

Prune the leaves.

$$V - E + F = (V - 1) - (E - 1) + F$$

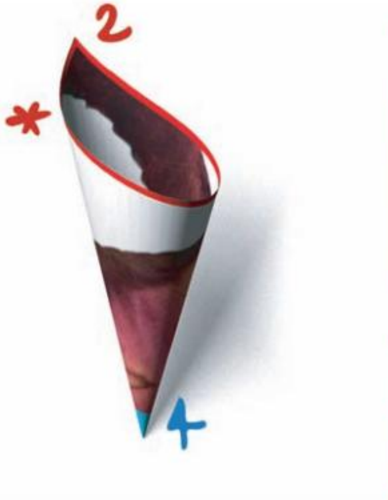
Eventually, only one vertex, and the outside face, will remain.

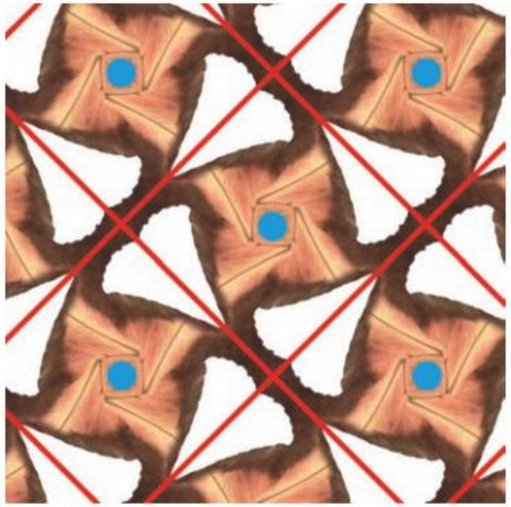
$$V - E + F$$
$$= 1 - 0 + 1 = 2$$

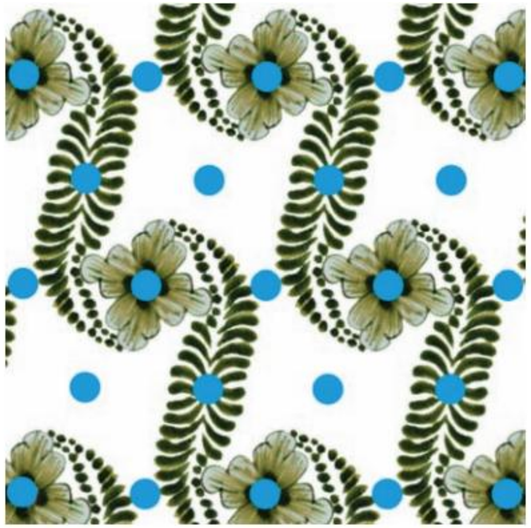
Orbifold

We can fold up a pattern into its smallest repeating unit. Its **orbit** under the group of symmetries is the whole pattern.

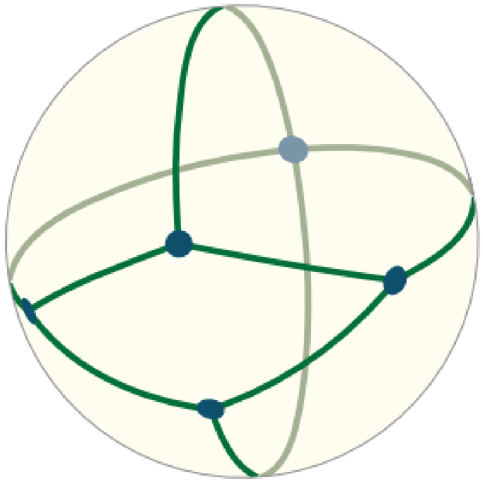
This allows us to represent any pattern as a surface.

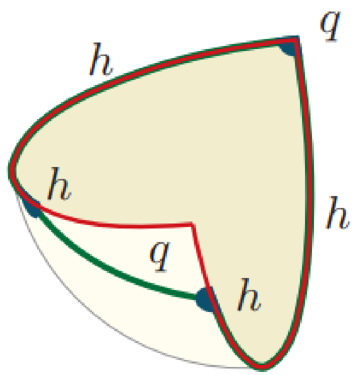






Effect on the Euler Characteristic

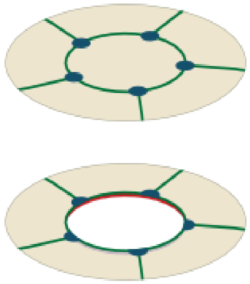




On the right we see the folded form of this map. Some of the vertices, edges, and faces have been halved (h) or quartered (q), so that we have

$$V = \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{5}{4},$$
$$E = 1 + \tfrac{1}{2} + \tfrac{1}{2} = 2,$$
$$F = 1 + \tfrac{1}{4} = \tfrac{5}{4}.$$

"Continuing in this fashion", one finds the change in Euler characteristic is

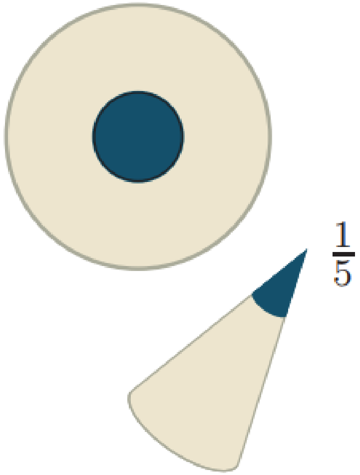


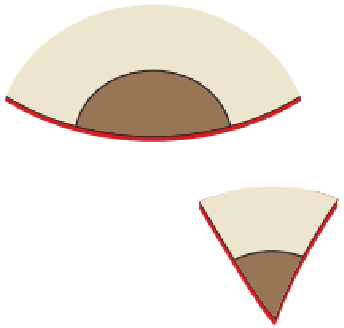
Punching a hole ($*$) decreases *char* by 1. Choose a map for which the "hole" is a single k-sided face. Then, removing it decreases F by 1 and V and E by $\frac{k}{2}$ (since vertices and edges around the hole get halved). Therefore, $V - E + F$ is reduced by $\frac{k}{2} - \frac{k}{2} + 1 = 1$.

The cost is the change in Euler Characteristic

Replacing an ordinary point by an N-fold cone point (**N**) decreases *char* by $\frac{N-1}{N}$. Choose a map for which the point is a vertex. Before the change, it contributes 1 to V; afterwards it contributes only $\frac{1}{N}$. The net change is

$$1 - \frac{1}{N} = \frac{N-1}{N}.$$





Replacing an ordinary boundary point by an N-fold corner point (N) decreases *char* by $\frac{N-1}{2N}$. Again, choose a map for which the given boundary point is a vertex. After the replacement, it will be $1/2N$ of a point, for a net change of $\frac{N-1}{2N}$.

To the plane...

We take a large disc of radius R from our infinitely repeating pattern, and project it onto the sphere.

The number of vertices, edges, and faces V, E, and F in the patch will be proportional to the number of copies $N = kR^2$, and the fractional portions v, e, f in the orbifold.

There is an error cR proportional to the perimeter.

$$ch = v - e + f < \left|\frac{V - E + F + cR}{N}\right| < \left|\frac{2 + cR}{kR^2}\right| \to 0$$

Adding symmetries performs an operation on the sphere/plane which changes its Euler characteristic.

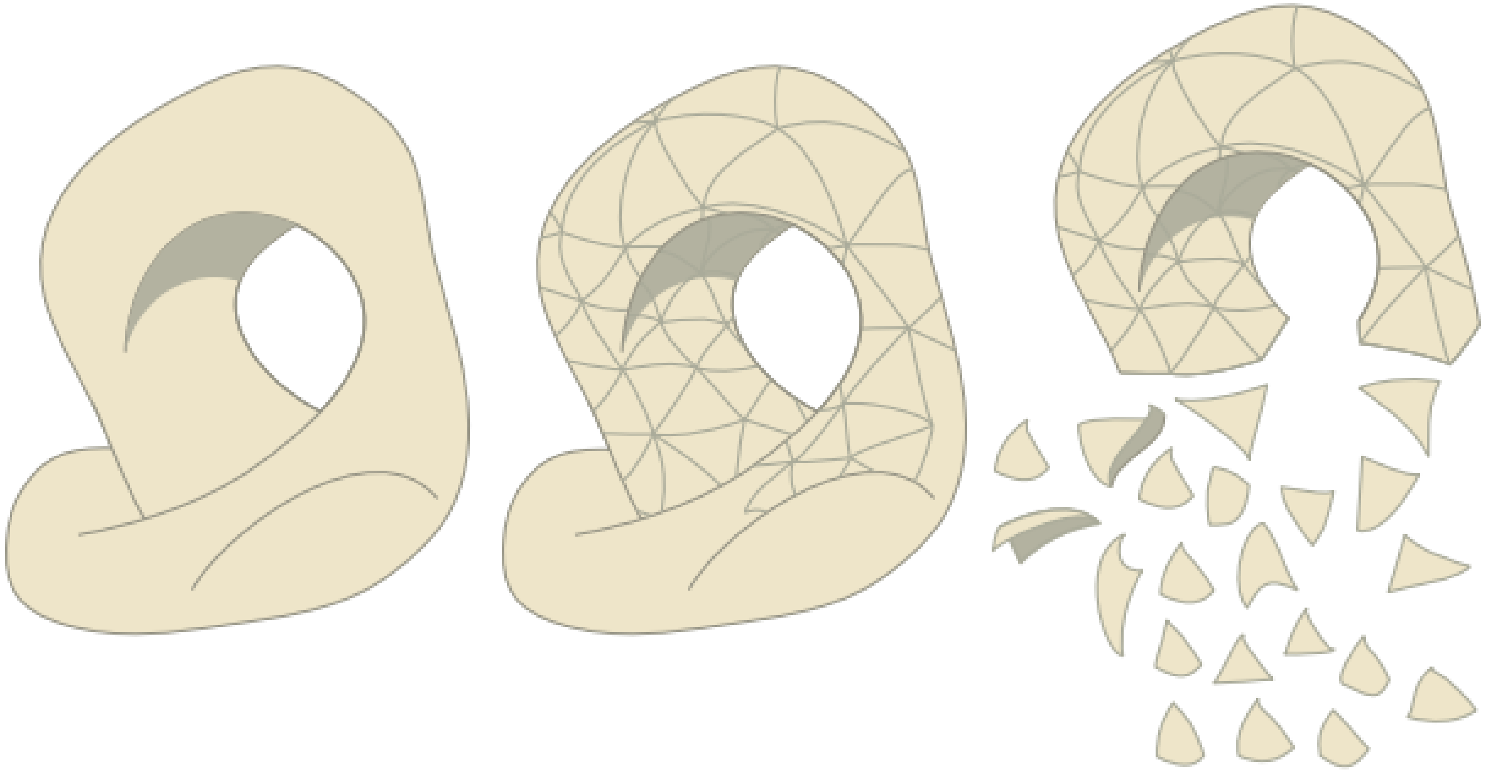
These are the costs from earlier

Why Part 2.

- We showed how these operations can be combined, but what if we missed some other operations?
- Each orbifold is a surface, so if we can show these operations can create all surfaces, then we will be able to describe any symmetry.

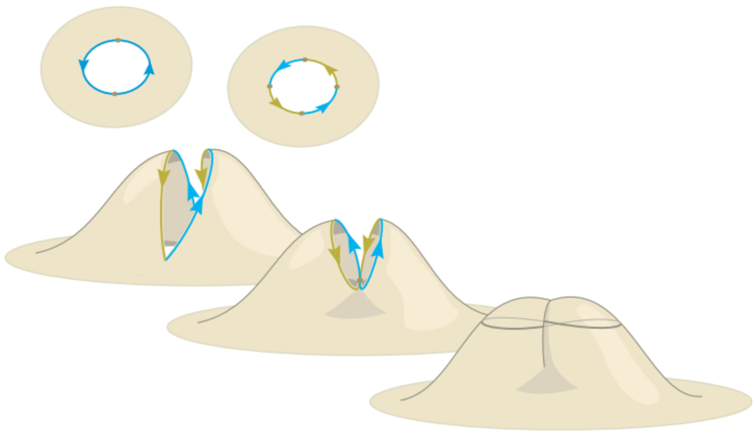
- It would also suffice to show that the operations we identified (reflection, rotation, translation, and glide reflection) generate all plane isometries (e.g. as in https://www.math.uchicago.edu/~may/VIGRE/VIGRE2009/REUPapers/Boswell.pdf)

Lemma 8.1 (Tidying Lemma) *Every surface is topologically equivalent to a "tidy" one, obtained from a collection of spheres by adding handles (○), holes (∗), crosscaps (×), and cross-handles (⊗).*

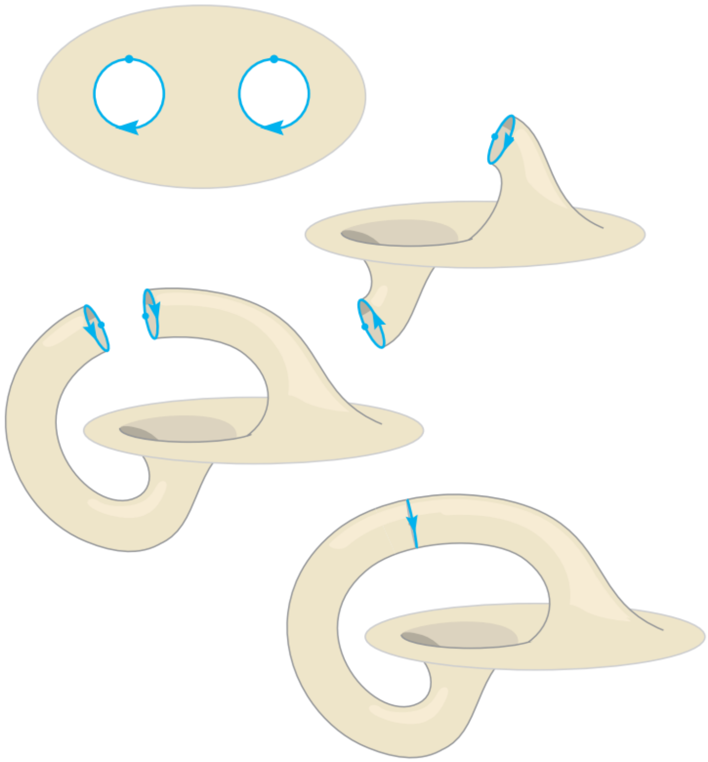


"It is a deep and difficult theorem, proved by Tibor Rado in 1925, that every compact 2-manifold is triangulable"

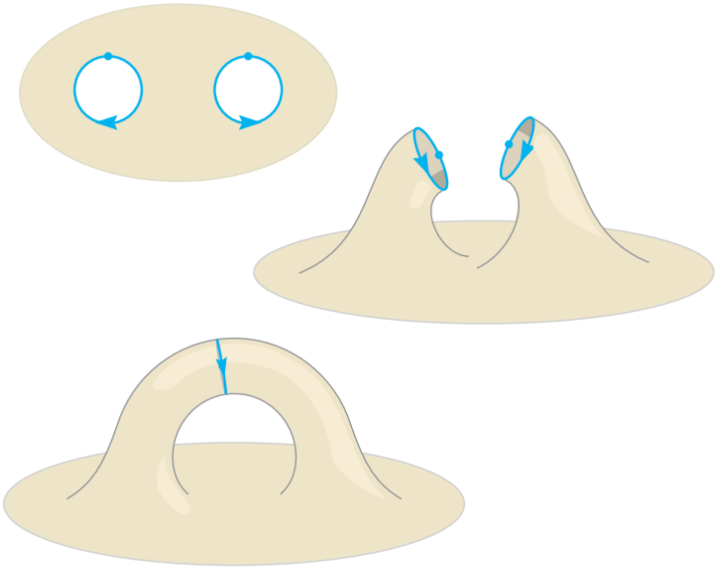
Theorem 8.2 (The Classification Theorem for Surfaces) *To obtain an arbitrary connected surface from a sphere, it suffices to add either handles or crosscaps and maybe to punch some holes, giving boundaries. So, the symbols* $\bigcirc^a *^b$ *and* $*^b \times^c$ *represent all possible surfaces.*



Zipping a crosscap.



Zipping a cross-handle.



Zipping a handle.

By topology, the four operations are sufficient to get every surface, so these operations can make any orbifold, and therefore any symmetry.

A symmetry by any other name…

Conway, Coxeter and crystallographic correspondence

Conway	o	××	*×	**	632	*632
Coxeter	$[\infty^+,2,\infty^+]$	$[(\infty,2)^+,\infty^+]$	$[\infty,2^+,\infty^+]$	$[\infty,2,\infty^+]$	$[6,3]^+$	$[6,3]$
Crystallographic	*p*1	*pg*	*cm*	*pm*	*p*6	*p*6*m*

Conway	333	*333	3*3	442	*442	4*2
Coxeter	$[3^{[3]}]^+$	$[3^{[3]}]$	$[3^+,6]$	$[4,4]^+$	$[4,4]$	$[4^+,4]$
Crystallographic	*p*3	*p*3*m*1	*p*31*m*	*p*4	*p*4*m*	*p*4*g*

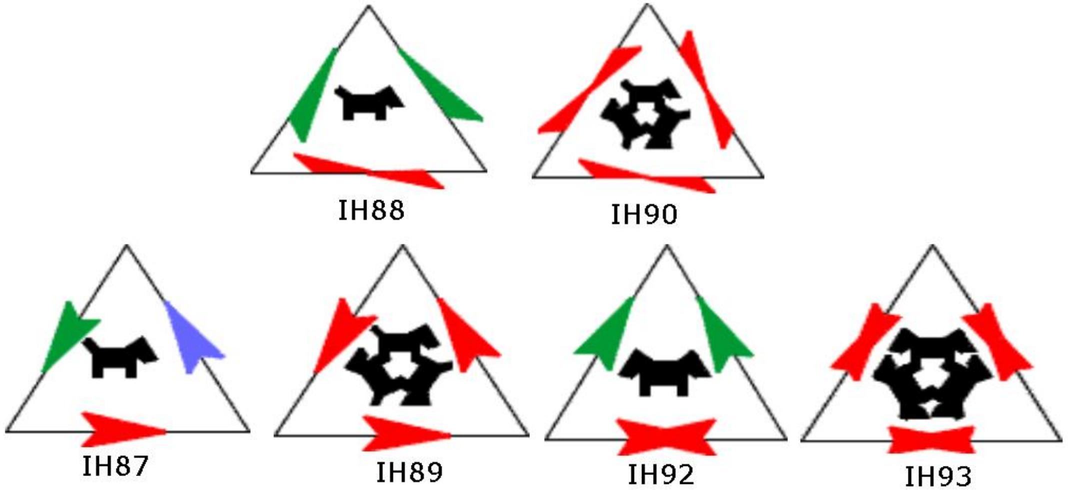
Conway	2222	22×	22*	*2222	2*22
Coxeter	$[\infty,2,\infty]^+$	$[((\infty,2)^+,(\infty,2)^+)]$	$[(\infty,2)^+,\infty]$	$[\infty,2,\infty]$	$[\infty,2^+,\infty]$
Crystallographic	*p*2	*pgg*	*pmg*	*pmm*	*cmm*

Source: wikipedia

In math, and life, taking the right perspective is key to solving challenging problems.

Reference

- This material came *The Symmetries of Things* by John H. Conway, Heidi Burgiel, and Chaim Goodman-Strauss
- There is lots more fun math there, including tiling on surfaces, colorings, hyperbolic symmetries, 3D tessellations, and polyhedra construction.
- The next time you notice a symmetry, try to identify it! If you enjoyed these ideas, share them with your friends / family.

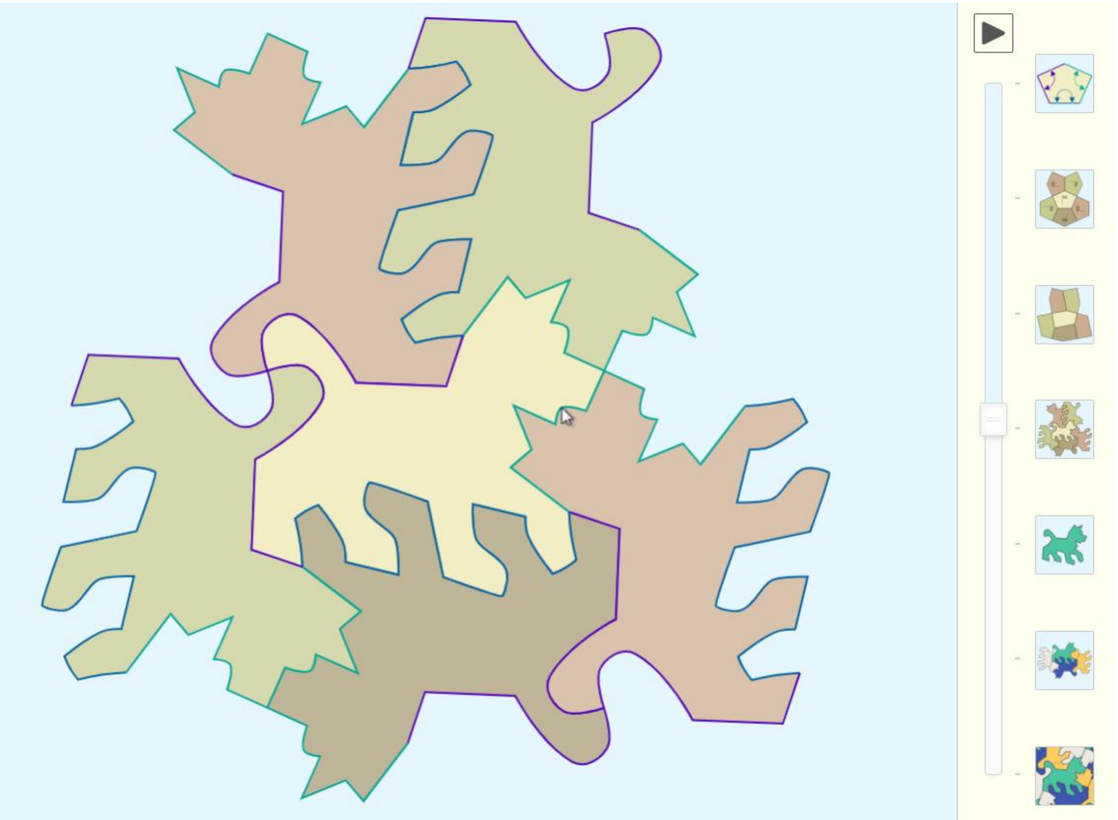


Tilings can be generated using templates

https://www.jaapsch.net/tilings/mclean/html/tutorial.html

This site animate the process

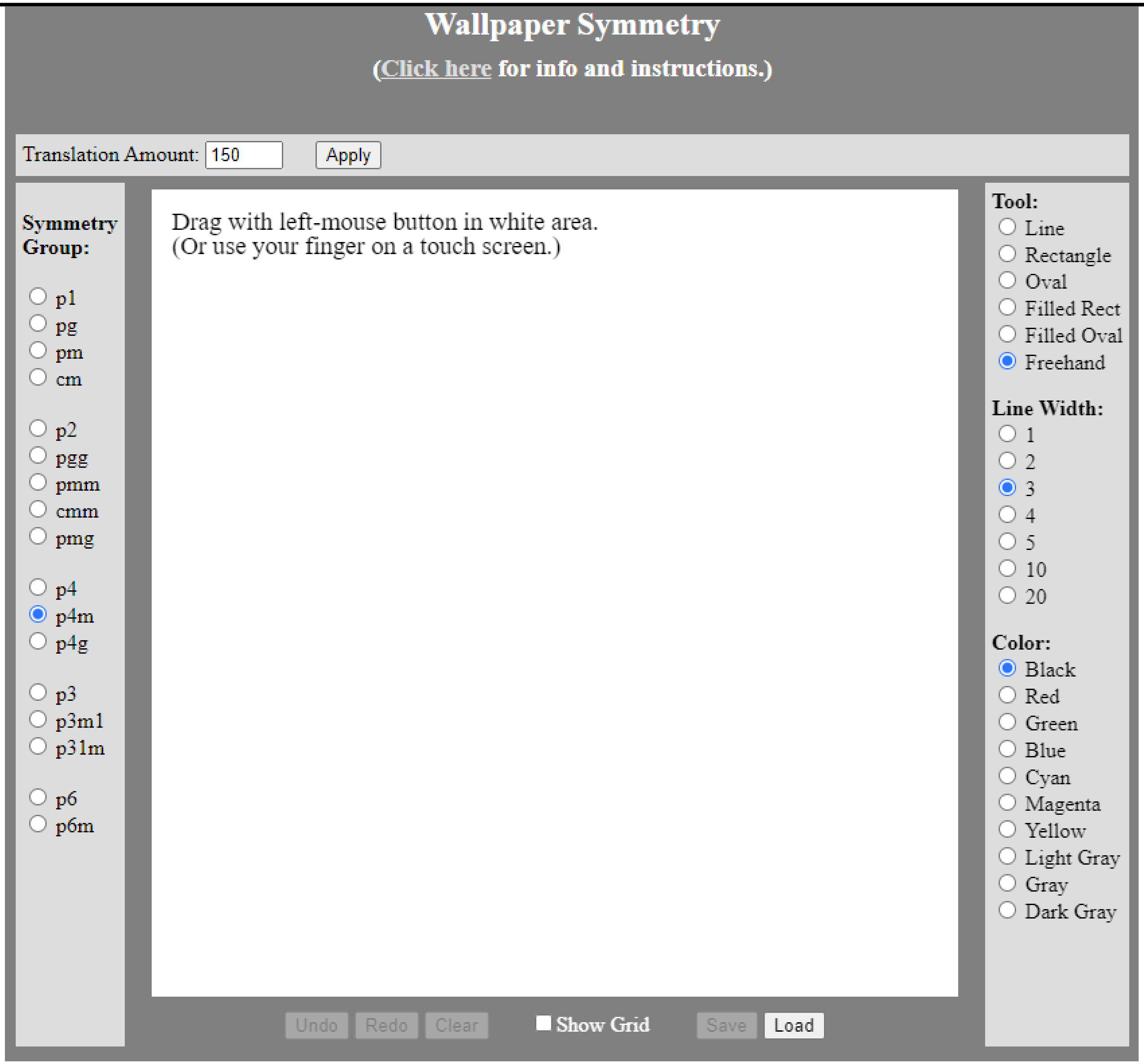
https://tiled.art/en/art/dogs10/



Live demo

This site lets you draw your own patterns in any of the 17 symmetries!

A lot of other programs do this too, but I like this one.



https://gallery.bridgesmathart.org/exhibitions/2021-bridges-conference-short-film-festival/alex-bannwarth

https://math.hws.edu/eck/js/symmetry/wallpaper.html

1. Orbifold notation is a way of describing symmetries.
2. Cost = change in Euler characteristic, allows for easy enumeration.
3. The classification of 2-manifolds shows this is complete.

What questions do you have?