👻 Spectral Activity 👻
Solutions

Strange things have been going on in $\mathbb{R}^4$, vectors have been moving when no one is looking! Investigators have determined that this is described by multiplication by the matrix

$$🎃 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

To better understand this phenomenon, you are looking for a basis that explains this multiplication simply. Your task is to find the "spectrum" of this matrix (the set of eigenvalues of a matrix) and put the spectres causing this mayhem to rest.

In this activity, we'll discover "diagonalization" and once again see the power of using different bases, as well as practicing abstraction by using the fact that math doesn't care what symbols we write to work with some more fun variables then A and $\vec{v}$.

1. Find the eigenvalues of this matrix, by determining the roots of the characteristic polynomial det(🎃 $-xI$).

Solution: This is $\begin{vmatrix} 1-x & 1 & 1 & 1 \\ 0 & 2-x & 1 & 1 \\ 0 & 0 & 3-x & 1 \\ 0 & 0 & 0 & 4-x \end{vmatrix} = (1-x)(2-x)(3-x)(4-x)$ since it is triangular, so has determinant the product of it's diagonal entries (this could also be found by cofactors using the last row). The eigenvalues are then the roots 1, 2, 3, and 4.

2. For each eigenvalue x, find a basis for the nullspace 🎃 $-xI$.

Solution: These are the corresponding matrices and a basis for each nullspace (this can be found by elimination and using the free columns, or by inspection).

$$N(A - 1I) = N\left(\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}\right) = \left\{ a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, ain\mathbb{R} \right\}$$

$$N(A - 1I) = N\left(\begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}\right) = \left\{ a \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, ain\mathbb{R} \right\}$$

$$N(A - 1I) = N\left(\begin{bmatrix} -2 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = \left\{ a \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, ain\mathbb{R} \right\}$$

$$N(A - 1I) = N\left(\begin{bmatrix} -3 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \left\{ a \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, ain\mathbb{R} \right\}$$

Therefore the corresponding eigenvectors are

$$\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

A witch comes along to help, and offers to perform a spell 🧙$^{-1}$ that will take any vector in the standard basis and express it in this new basis of eigenvectors, all through the magic of matrix multiplication! They also provide the reverse spell 🧙 to transform it back:

$$\left(🧙\right)^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad 🧙 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Follow the steps below to see what $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ becomes when multiplied by 🎃.

3. Multiply $\vec{v}$ by 🧙$^{-1}$, the components of the resulting vector are the coefficients needed to express $\vec{v}$ as a combination of the eigenvectors of 🎃. Write this linear combination explicitly as $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4v_4$.

Solution: This is $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}$ so $\vec{v} = -1\vec{v}_1 - 1\vec{v}_2 - 1\vec{v}_3 + 4\vec{v}_4$.

4. Multiply this linear combination by 🎃, using linearity and the fact that 🎃$\vec{v})i = \lambda\vec{v}_i$ for any eigenvector. Add them together to get the value 🎃$\vec{v}$. If you want, you can compute this directly by matrix vector multiplication to check they match.

Solution: We have

$$🎃\,\vec{v} = 🎃(-1\vec{v}_1 - 1\vec{v}_2 - 1\vec{v}_3 + 4\vec{v}_4)$$

$$= -1\,🎃\,\vec{v}_1 - 1\,🎃\,\vec{v}_2 - 1\,🎃\,\vec{v}_3 + 4\,🎃\,\vec{v}_4 = -1 * 1\vec{v}_1 - 1 * 2\vec{v}_2 - 1 * 3\vec{v}_3 + 4 * 4\vec{v}_4$$

$$= -1\vec{v}_1 - 2\vec{v}_2 - 3\vec{v}_3 + 16\vec{v}_4 = -1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 2\vec{v}_2\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + 16\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \\ 13 \\ 16 \end{bmatrix}$$

Alternatively, let $\vec{w} =$ 🧙$^{-1}\,\vec{v}$ be the vector whose components are the coefficients of that linear combination.

5. Find a diagonal matrix $\skull$ that scales each coefficient by the appropriate eigenvalue (as done in the previous step). This way $\skull\vec{w}$ gives the coefficients of $\pumpkin\vec{v}$ when expressed as a linear combination of the eigenvectors (the first part of the previous question).

> **Solution:** If we put the eigenvalues on the diagonal we have $\skull = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix}$ and so $\skull\vec{w} = \begin{bmatrix} -1 \\ -2 \\ -3 \\ 16 \end{bmatrix}$.

6. Multiply the vector $\skull\vec{w}$ by the matrix $\mathwitch$ to get the result back in the standard basis. This should agree with part 4.

> **Solution:** This is $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \\ 13 \\ 16 \end{bmatrix}$

We showed $\pumpkin\vec{v} = \mathwitch\skull(\mathwitch)^{-1}\vec{v}$ for the given vector $\vec{v}$. This same approach works for any vector, so we can say

$$\pumpkin = \mathwitch\,\skull\,(\mathwitch)^{-1}$$

using the "skeleton" of eigenvalues $\skull$ found in 5 and the witch's spells. This is the "diagonalization" of a matrix.

7. Write down this equation explicitly using each matrix we have in this case.

> **Solution:** For these matrices, we have
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> $$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

8. **<u>Bonus</u>**: What do you notice about the columns of ? This explains how the witch's spell works!

Solution: The columns of  are the eigenvectors of .