MATH 22: Activity: The rank and solving $A\vec{x} = \vec{b}$ - Solutions

Today's we'll derive the possible number of solutions to $A\vec{x} = \vec{b}$ based on the rank and dimensions of A. Let A be any $m \times n$ matrix with rank r. That is, A has m rows and n columns, of which r are pivots and $n - r$ are free. Since r can only either be less than or equal to n, and the same for m, there are four distinct cases, listed in the first column below.

Match the following statements and use them to mark the possible numbers of solutions in each case given in the table at the bottom of the page.

$r < m$, so there are rows without pivots	The column space is not the whole space	Some $\vec{b}$ has zero solutions
$r = m$, so every row has a pivot	The column space is the whole space	Every $\vec{b}$ has at least one solution
$r < n$, so there are free columns	The Nullspace is nontrivial	Whenever $\vec{b}$ has one solution, it has infinitely many, so zero solutions is impossible
$r = n$	The nullspace is trivial	It is impossible to have infinitely many solutions

	Possible solutions		
cases	0	1	infinitely many
$r = m, r = n$		✓	
$r = m, r < n$			✓
$r < m, r = n$	✓	✓	
$r < m, r < n$	✓		✓

To better understand the column and null spaces, write down a simple matrix A for each case and find values of $\vec{b}$ with each possible number of solutions.

$r = m$, $r = n$: Some examples are $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, or any vector, has exactly one solution.

$r = m$, $r < n$: Some examples are $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has infinitely many solutions.

$r < m$, $r = n$: Some examples are $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ or $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ have zero and one solutions, respectively.

$r < m$, $r < n$: Some examples are $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ have infinitely many and zero solutions, respectively.