Probability Paradoxes,

Eleven Enlightening Examples

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Introduction

We think of probability as straightforward and unambiguous, but this is because we learn it through well defined rules and mainly apply it to easy situations.

Throughout history, several "paradoxes" have been found, situations with conflicting analyses. Today, we'll examine eleven of these to see why probability is not as simple as we might think.

Birthday Paradox

In a group of n people, what are the odds that two share a birthday (assuming they are independent and uniformly distributed, and no leap days)?

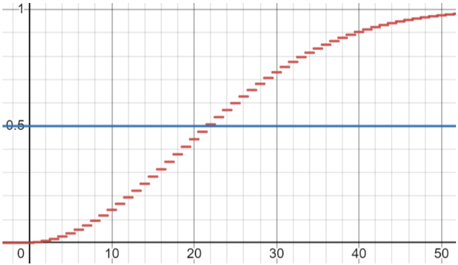
For what n is this at least half?

Since there are 365 days in a year, you might expect the probability to be around $n/365$, but in fact this is far too low.

The correct calculation is

$$P_n = 1 - \frac{365}{365} \cdot \frac{365-1}{365} \cdot \frac{365-2}{365} \cdot \frac{365-3}{365} \cdots \frac{365-n}{365}$$

which looks like



Two Child Problem

Mr. Smith has two children (whose genders are independent and uniformly distributed).

If you learn one of them is a girl, what is the probability that the other child is a girl?

What would this probability be if instead your learned that the older child is a girl?

"The older child is a girl" → "At least one child is a girl"

Older	Younger
G	G
G	B
B	G
B	B

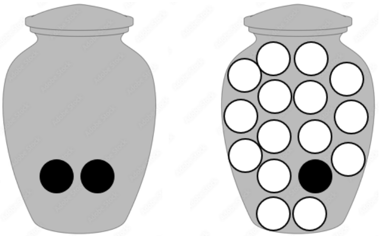
Older	Younger
G	G
G	B
B	G
B	B

Betrand's Box Paradox

You are given one of these urns uniformly at random, and draw a ball uniformly at random. If it is black, what is the probability you were given the BB-urn?

Many people say 1/2, as you can rule out the WW urn, and the other two were equally likely.

Drawn ball	Undrawn ball
W1	W2
W2	W1
W	B
B	W
B1	B2
B2	B1



An extreme example makes this more intuitive, consider the two urns above, now it makes more sense why a black ball is better evidence for the BB urn.

Bertrand's Chord Paradox

What is the probability a random chord has length at least as long as the side of an inscribed equilateral triangle?

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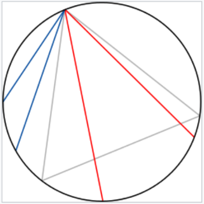
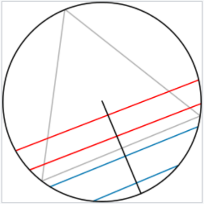
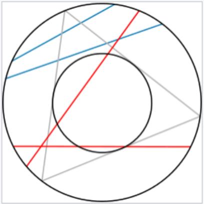


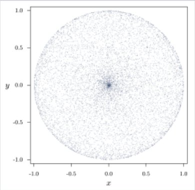
Random chords, selection method 1; red = longer than triangle side, blue = shorter



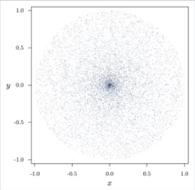
Random chords, selection method 2



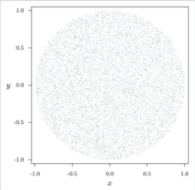
Random chords, selection method 3



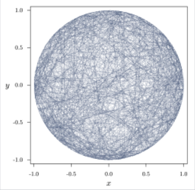
Midpoints of the chords chosen at random using method 1



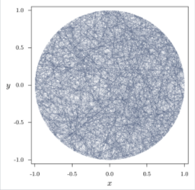
Midpoints of the chords chosen at random using method 2



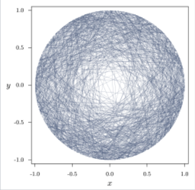
Midpoints of the chords chosen at random using method 3



Chords chosen at random, method 1



Chords chosen at random, method 2



Chords chosen at random, method 3

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St. Petersberg Paradox

You have an offer to play the following game. A pot of money starts with one dollar, and a fair coin is repeatedly flipped:

- Each time it lands heads, the amount of money is doubled.
- Once it lands tails, the game ends and you win all the money.

How much should you be willing to pay to play this game?

Well, one guess might be it's expected value,

$$E = \frac{1}{2}1 + \frac{1}{4}2 + \frac{1}{8}4 + \frac{1}{16}8 + \dots$$
$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$
$$= \infty$$

so you should be willing to pay any amount!

Despite this, note you only have a $1/n$ chance of breaking even on a bet of n dollars, as to win $n \approx 2^k$ in this game, you need to flip at least k heads, which happens with probability $(1/2)^k \approx 1/n$.

One solution is to note that our space isn't really infinite. If there is only so much money you could really win, for example, if the maximum payout was $228 Billion (net worth of Elon Musk), then the expected value would only be

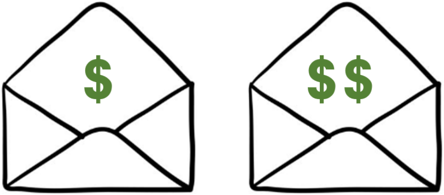
$$\frac{1}{2}\log_2(228 \text{ billion}) \approx 18.87$$

Another is to assume people's utilities, the value they derive from something, are not linear, then the series could converge.

Or perhaps people simply disregard any sufficiently rare event.

Two Envelope Paradox

You are given the choice between two envelopes, one with twice as much money in it as the other.



You choose an envelope and look inside, but then are given the chance to switch and take the other envelope instead. Should you do it?

It seems like there is no reason to, as you had no information to base your initial choice on, and you can't tell if the value you saw was the larger or smaller one.

However, if x represents the amount of money in your envelope, then

$$E[other] = \frac{1}{2}\frac{x}{2} + \frac{1}{2}2x = 1.25x > x$$

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The issue is that this calculates expected value wrong. Suppose the envelopes contain y and $2y$. The average change in wealth is then

$$E[switch] = \frac{1}{2}y + \frac{1}{2}(-y) = 0$$

That is, there is no benefit to switching, as expected.

Non-Transitive Dice

Three fair dice are labelled as follows. Two players each choose a die, and the one who rolls a higher number wins.

Red	3	3	3	3	3	6
Blue	2	2	2	5	5	5
Green	1	4	4	4	4	4

All the dice have the same expected value as a normal die, so you'd expect there to be no winning strategy.

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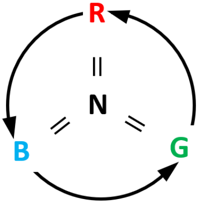

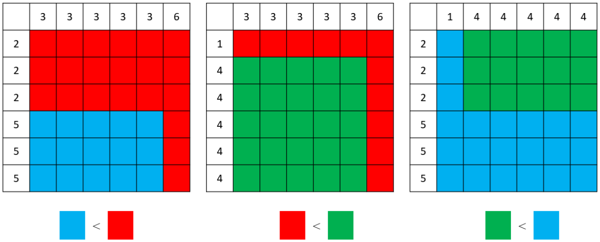
3 3 3 3 3 6
2 2 2 5 5 5
3 3 3 3 3 6
1 4 4 4 4 4
1 4 4 4 4 4
2 2 2 5 5 5
<
<
<

	3	3	3	3	3	6
2						
2						
2						
5						
5						
5						

	3	3	3	3	3	6
1						
4						
4						
4						
4						
4						

	1	4	4	4	4	4
2						
2						
2						
5						
5						
5						

Even crazier, the order reverses with two rolls!

A set of seven dice was found by Oskar van Deventer where for any pair of dice, one can choose a die that beats both on average.

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Base Rate Fallacy

Suppose you take a test that is correctly identifies a disease 95% of the time, the correct positive (and negative) rate.

If this disease occurs in 1% of the population, what is the probability someone with a positive test truly has the disease?

Most people would say 95%, since the test is correct that amount of the time, but this fails to take into account the fact that the disease is rare, it has a small base rate.

The proper calculation is to see the proportion of true positives, out of total positives, true positive plus false positive:

$$\frac{0.01 * 0.99}{0.01 * 0.99 + 0.99 * 0.95} \approx 16.2\%$$

Since the initial calculation didn't factor in the low "base rate" of the disease, this effect is called the Base Rate Fallacy.

This same issue occurs any other time there is a low base rate, e.g. identifying criminals or terrorists, say by DNA testing.

Another issue can occur, called the Prosecutor's Fallacy, whereby so many searches are performed, one randomly ends up positive.

There are lots of interesting examples of this and related effects in the book "Math on Trial" by Leila Schneps and Coralie Colmez.

Simpson's Paradox

In 1973, the University of California Berkely Graduate School was sued for sex bias in admissions. The below table summarizes the key evidence.

	Men	Women
Applied	2590	1835
Admitted	1189	554
Percent	46%	30%

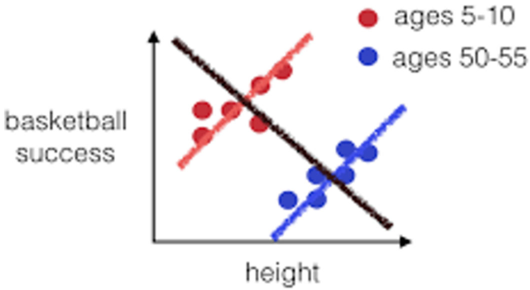
Despite this, the acceptance rate of women was actually higher in many departments. How can this be?

Department	Men			Women		
	applicants	admitted	percent	applicants	admitted	percent
A	825	511	62%	108	88	82%
B	560	352	63%	25	17	68%
C	325	120	37%	593	201	34%
D	417	137	33%	375	131	35%
E	191	53	28%	393	94	24%
F	272	16	6%	341	23	7%
Total	2590	1189	46%	1835	554	30%

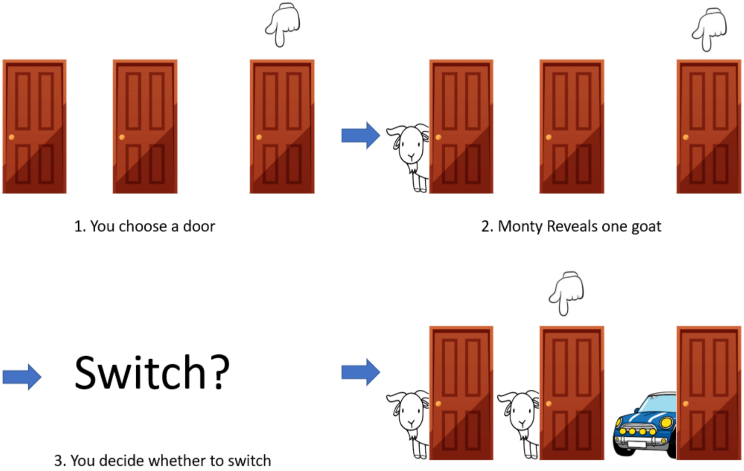
Men applied more often to departments with higher acceptance rates, skewing their average.

Another example: A worse surgeon appears better because they perform more routine surgeries.

This can also happen with continuous variables:

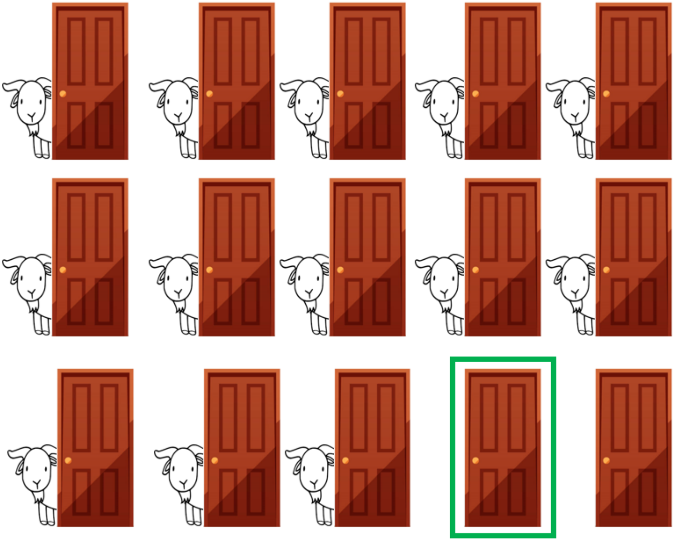


Monty Hall Paradox



This problem was very controversial, rising to prominence after a 1990 magazine column by Marilyn vos Savant. Despite giving the correct answer, many people wrote angry letters saying she was wrong.

The key idea is that you are guaranteed to win by switching if the car was behind either of the two doors you didn't initially pick. Since your initial choice had no information, this probability is $2/3$.



Sleeping Beauty Problem

Sleeping beauty participates in the following study.

- Sunday - SB is put into a deep sleep with a memory erasing drug, and a coin is flipped.
- Monday - SB is woken and put back to sleep.
- Tuesday - If the coin is tails, SB is woken and put back to sleep.
- Wednesday - SB is woken and the experiment ends.

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Suppose when Sleeping Beauty is woken up, she is asked what is the probability that the coin came up heads. What should she say?

There were two main camps on this.

The answer is 1/2 - no new info, she knew she was going to be woken up!

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The answer is 1/3 - Being awake is like drawing a black ball, where one has equal chances of being given a BB or BW urn:

	Monday	Tuesday
Coin Heads	Awake	Asleep
Coin Tails	Awake	Awake

Questions?